On verification of a superpopulation model

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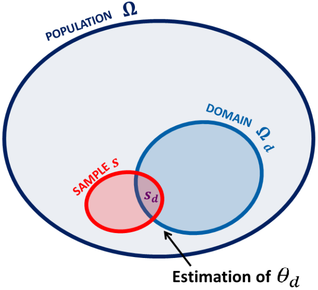
The aims of studies

- An analysis of the possibility of using **parametric bootstrap test** for testing **dependence between random effects**.
- A **comparison** in a simulation study the **properties** of this significance test with one of the **classic test**.
- Analysis taking into account the problem of:
  - **model misspecyfications**,
  - **non-normality** of **random effects**.

Introduction - Small Area Estimation

Small area - domain for which we cannot obtain direct estimates with adequate precision (Rao, 2003, p. 2).



One of the main approache in small area estimation - **model-based approach**. A **superpopultion model** as a **source of randomness** in this approach.

Linear mixed model

The general linear mixed model (cf. Jiang, 2007, pp. 1-2):

$$\mathbf{Y} = \mathbf{X}\beta + \mathbf{Z}\mathbf{u} + \epsilon \tag{1}$$

where:

- $\mathbf{Y}$ – the random vector of values of the dependent variable;
- $\mathbf{X}$, $\mathbf{Z}$ – known matrices of auxiliary variables;
- β – the vector of unknown parameters.

Random effects $\mathbf{v}$ and stochastic disturbance ϵ are independently distributed with variance-covariance matrices denoted by $\mathbf{G}(\delta)$ and $\mathbf{R}(\delta)$, where δ is a vector of variance components.

Linear mixed model with correlation of random effects (1)

We can consider correlation of random effects as two cases:

- **autocorrelation** of random effect
  - simultaneously autoregressive process of random effect in estimation the annual per-capita mean income in Tuscany (Pratesi and Salvati, 2008);
  - normal mixed autoregressive moving average process of random effects (Tiao and Ali, 1971);
  - serial correlation of random effect in estimation of production for Japanes chemical industry (Skoglund and Karlsson, 2001);
- **correlation between** random effects
  - to estimation plasma concentration some drug by nonlinear mixed effects model (Dumont C. et. al., 2014);
  - to analyze the health care costs at the end of life (Menec et al. 2004).

Linear mixed model with correlation of random effects (2)

$$Y_{id} = (\beta_1 + v_1)x_{id} + \beta_0 + v_2 + e_{id} \tag{2}$$

where:

- Y_{id} – values of the dependent variable;
- x_{id} – values of the auxiliary variables;
- β_0, β_1 – unknown parameters

  and
- $\mathbf{G}(\boldsymbol{\delta}) = D^2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \sigma_{v_1}^2 & \rho\sigma_{v_1}\sigma_{v_2} \\ \rho\sigma_{v_1}\sigma_{v_2} & \sigma_{v_2}^2 \end{bmatrix}$
- $\mathbf{R}(\boldsymbol{\delta}) = \sigma_e^2 \mathbf{I}_n$

Classic test - Likelihood Ratio Test

We considered the following **hypotheses**:

$H_0 : \rho = 0$
$H_1 : \rho \neq 0$

The test statistic is given by:

$$2log\left(\frac{L_2}{L_1}\right) = 2\left[log(L_2) - log(L_1)\right] \tag{3}$$

where:

- L_1 - the likelihood for more general model;
- L_2 - the likelihood for the restricted model.

Under the hypothesis H_0 this test statistic has asymptotic χ^2 distribution with $k_2 - k_1$ degrees of freedom (where k_1 and k_2 are the number of model parameters) (Pinheiro and Bates, 2000, p. 83).

Monte Carlo Test

Idea of the **Monte Carlo test** was first described by Bartlett (1963) and Barnard (1963).
This test based on the comparison of original data with random samples, which are generated under the null hypothesis (Hope, 1968, p. 582).
It should be noted similarity of developed in 1980's **parametric bootstrap approximation** to the procedure of **Monte Carlo test** (Zhu, 2005, p. 2).

Parametric bootstrap test (1)

The prametric bootstrap method finds application in many areas i.a.:

- estimation of the **MSE** (Gonzales-Manteiga et. al., 2008; Butar and Lahiri, 2003);
- approximation of the distribution **LRT statistic** (Shaw and Geyer, 1997);
- approximation of the distribution **EBLUP** (Chatterje et. al., 2008);
- Bayesian inference (Efron, 2012).

Parametric bootstrap test (2) - testing procedure

1. Estimation ρ for the original data set (denoted by ρ_0);
2. B-times:
   a) generating data under hypothesis H_0;
   b) estimation ρ_{*b} according to the model with correlated random effects;
3. Calculate p-value as $p = \frac{1 + b: |\rho_{*b}| > |\rho_0|}{1+B}$.

Simulation study

- **assumptions**
- **results of part I of the simulation study**
- **results of part II of the simulation study**
- **conclusions**

Simulation study (1)

Data set:

- data Särndal et al (1992);
- population elements – counties in Sweden ($N = 284$);
- y - revenues from municipal taxation (in millions of kronor);
- x - population (in thousands);
- domains according to the cluster indicator ($D = 50$).

Simulation study (2) - choice of the model

$$Y_{id} = \beta_1 x_{id} + \beta_0 + e_{id} \quad (4)$$

$$Y_{id} = \beta_1 x_{id} + \beta_0 + v_d + e_{id} \quad (5)$$

$$Y_{id} = (\beta_1 + v_d) x_{id} + \beta_0 + e_{id} \quad (6)$$

$$Y_{id} = (\beta_1 + v_{1d}) x_{id} + \beta_0 + v_{2d} + e_{id} \quad (7)$$

$$Y_{id} = (\beta_1 + v_{1d*}) x_{id} + \beta_0 + v_{2d*} + e_{id} \quad (8)$$

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Simulation study (3)

- We compare the properties of two test - **Likelihood Ratio Test** and **parametric bootstrap test**:
- We considered the problem of **model misspecification** and **non-normality** of random effects;
- We divide simulation study into part:
  - **part I** - probability of **type I** errors;
  - **part II** - probability of **type II** errors;
- The number of MC iterations equals 1.000 and the number of bootstrap iterations – 200;
- The simulation study was prepared using R language (R Development Core Team, 2018);

The problem of non-normality of random effects (1)

In part I of the simulation:

- **normal** distribution with expected value equal 0;
- shifted **exponential** distribution with expected value equal 0;
- shifted **gamma** distribution with expected value equal 0 and coefficient of asymmetry equal 4

and variance computed based on real data.

The problem of non-normality of random effects (2)

In part II of the simulation:

- **multivariate normal** distribution with expected values equal 0 and $\rho = \{-0.9; -0.6, -0.3; 0.3; 0.6; 0.9\}$;
- **normal copula** with $\rho = \{-0.9; -0.6, -0.3; 0.3; 0.6; 0.9\}$ and marginal distributions:
  - shifted **exponential** distribution;
  - shifted **gamma** distribution.
- **t copula** with $\rho = \{-0.9; -0.6, -0.3; 0.3; 0.6; 0.9\}$, $df = 3$ and marginal distributions:
  - shifted **exponential** distribution;
  - shifted **gamma** distribution.

The problem of non-normality of random effects - copula function (1)

$$C(u) : [0,1]^n \to [0,1]$$

$$H(x,y) = C\left(F_1(x), F_2(x)\right)$$

(Sklar, 1959)

Gaussian copula

$$C_\rho(u_1, v_2) = \Phi_\rho\left(F_1(u_1), F_2(u_2)\right)$$

t copula

$$C_{\rho,v}(u_1, v_2) = T_{\rho,v}\left(F_1(u_1), F_2(u_2)\right)$$

(cf. Nelsen, 1999, pp. 46-47)

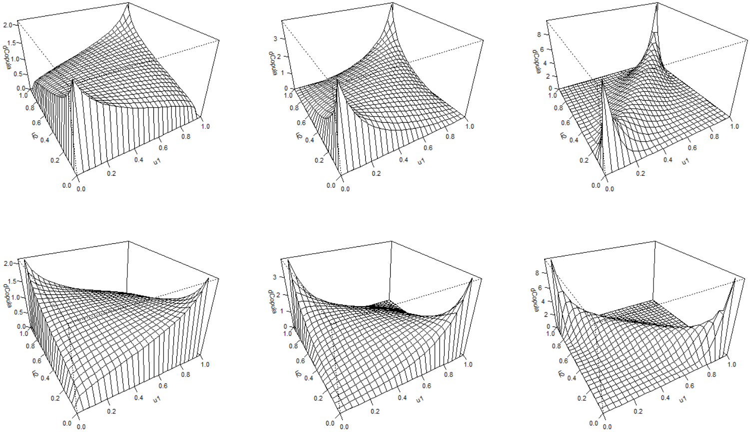
The problem of non-normality of random effects - copula function (2)

Copula functions have applications i.a. in:

- modeling the relationship between mortality and losses of the insurer (Frees et al., 1996; Frees et al., 2005);
- modeling insolvency risk and risk management (Embrechts et al., 2003);
- portfolio optimization (Patton, 2004);
- the processes of multidimensional control and hydrological modeling (Yan, 2006; Genest and Faure, 2007);
- twodimensional storm surges analysis (Ciupak and Rokiciński, 2011);
- modeling the dependence of the prices of futures contracts on agricultural products (Hołda and Malik, 2013).

The problem of non-normality of random effects - copula function (3)

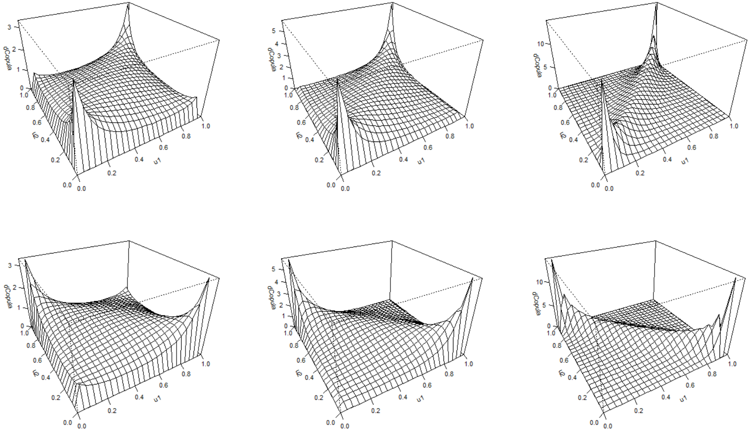
Gaussian copula



(Source: own elaboration)

The problem of non-normality of random effects - copula function (4)

t copula



(Source: own elaboration)

Results - part I

Table 1. The values of probabilities of type I errors.

norm		exp		gam	
LRT	**PB**	**LRT**	**PB**	**LRT**	**PB**
0.042	0.036	0.053	0.042	0.050	0.044

(Source: own elaboration)

Table 2. The values of probabilities of type II errors.

	mvnorm		normcop.exp		tcop.exp		normcop.gam		tcop.gam	
ρ	**LRT**	**PB**	**LRT**	**PB**	**LRT**	**PB**	**LRT**	**PB**	**LRT**	**PB**
-0.9	0.000	0.000	0.000	0.000	0.002	0.001	0.476	0.516	0.521	0.556
-0.6	0.006	0.006	0.029	0.032	0.122	0.127	0.717	0.744	0.800	0.831
-0.3	0.442	0.447	0.626	0.644	0.653	0.662	0.952	0.957	0.901	0.898
0.3	0.397	0.414	0.549	0.553	0.459	0.467	0.691	0.693	0.460	0.461
0.6	0.001	0.004	0.025	0.022	0.041	0.043	0.151	0.156	0.144	0.147
0.9	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.000

(Source: own elaboration)

Conclusions - part I

In the simulation study we show that for the considered real data and model (6):

- **parametric bootstrap test** gives lower values of the probability of **type I** errors than **Likelihood Ratio Test**;
- considered tests are **quite robust** on **non-normality** of random effects if assumption about the lack of correlation is fulfilled;

Conclusions - part II

Obtained results show that:

- for **moderate** and **strong** correlation considered tests have good properties - low values of the probability of **type II** errors, except some cases with marginal gamma distribution of the random effects;
- if random effects have **multivariate normal distribution** the power of the considered tests is higher than **0.5** even for the weak correlation.

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Thank you
for Your attention